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EDITED BY

F. N. COLE VIRGIL SNYDER J. W. YOUNG
ALEXANDER ZIWET D. E. SMITH
T. LEVI-CIVITA R. C. ARCHIBALD

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

ON A SMALL VARIATION WHICH RENDERS A LINEAR DIFFERENTIAL SYSTEM INCOMPATIBLE.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, April 25, 1914.)

LET us consider a homogeneous linear differential expression of the nth order*

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

whose coefficients are continuous functions of the real variable x in a closed interval ab. We suppose that l_n does not vanish in this interval. We consider the $2n$ quantities

$$u(a), u'(a), \cdots, u^{[n-1]}(a); \qquad u(b), u'(b), \cdots, u^{[n-1]}(b)$$

and form n linearly independent linear forms in them, $U_1(u)$, $\cdots$, $U_n(u)$, with constant coefficients.

Consider now the homogeneous linear differential system

$$(1) \qquad L(u) = 0, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n).$$

This system is said to have k-fold compatibility if there are k and only k linearly independent functions which satisfy it. It is well known and immediately obvious that, if y_1, $\cdots$, y_n is any fundamental system of the equation $L(u) = 0$, a necessary and sufficient condition for k-fold compatibility is that the rank of the matrix

* No additional difficulties would be introduced if we considered the more general expressions treated in my paper, *Transactions*, vol. 14 (1913), p. 403. See in particular the latter part of § 3.

$$(2)\qquad \begin{pmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ U_n(y_1) & \cdots & U_n(y_n) \end{pmatrix}$$

be $n-k$. Since all the elements, and hence all the determinants, of this matrix will be only slightly changed by a small variation of the coefficients of the system (1) (provided that, as is obviously possible, the y_i's and their first $n-1$ derivatives are allowed to vary only slightly) we immediately infer the following important result:

THEOREM I. *If the system* (1) *has k-fold compatibility, it has no higher order of compatibility after any variation of its coefficients which is uniformly sufficiently small in ab.**

While this theorem tells us that *no* very small variation will *raise* the order of compatibility, the main result to be established in this paper refers to the possibility of *lowering* the order of compatibility, and here we shall prove not merely that there always exist arbitrarily small variations which render the system incompatible (i. e., reduce its order of compatibility to zero) but that a variation of a very simple and important type will have this effect; namely a *real* variation of the coefficient l_0 alone (so that the conditions $U_i = 0$ are not varied) and, indeed, a variation which is everywhere positive, or, what is not essentially different, everywhere negative. The proof will depend on certain preliminary lemmas.

Let us suppose that the system (1) has k-fold compatibility, and, as a matter of notation, let us suppose that the $(n-k)$-rowed determinant in the upper left-hand corner of the matrix (2) is not zero. Then every solution of the equation $L(u) = 0$ which satisfies the first $n-k$ conditions $U_i = 0$ will also satisfy the remaining conditions. Such a function is given by the determinant

$$(3)\qquad \begin{vmatrix} y_1 & \cdots & y_{n-k} & c_1 y_{n-k+1} & + \cdots + & c_k y_n \\ U_1(y_1) & \cdots & U_1(y_{n-k}) & c_1 U_1(y_{n-k+1}) & + \cdots + & c_k U_1(y_n) \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ U_{n-k}(y_1) & \cdots & U_{n-k}(y_{n-k}) & c_1 U_{n-k}(y_{n-k+1}) & + \cdots + & c_k U_{n-k}(y_n) \end{vmatrix}$$

* The special case $k = 0$ of this theorem tells us that if the system (1) is incompatible, it remains so after every variation of its coefficients which is uniformly sufficiently small.

Moreover this determinant vanishes identically only when $c_1, \cdots, c_k$ are all zero, since otherwise $y_1, \cdots, y_n$ would be linearly dependent. Consequently the formula (3) gives a linear family of solutions of the system (1) whose bases consist of just k functions, so that, since by hypothesis (1) has k-fold compatibility, (3) gives its general solution.

Let us now suppose that the coefficients of $L(u)$ are continuous functions of (x, λ) and that the coefficients of the U_i's are continuous functions of λ; and that when $\lambda = \lambda_0$ and for a certain neighborhood of this value the system (1) has just k-fold compatibility. If we arrange the notation so that when $\lambda = \lambda_0$ the $(n - k)$-rowed determinant in the upper left-hand corner of (2) is not zero, we can take the neighborhood of λ_0 so small that this same determinant does not vanish in this neighborhood, provided that, as is surely possible, $y_1, \cdots, y_n$ are so chosen that they and their first $n - 1$ derivatives are continuous functions of (x, λ). Then (3) gives the general solution of the system (1) for all values of λ in a certain neighborhood of λ_0, and it is clear that for any special determination of the c_i's, either as constants or as continuous functions of λ, the function (3) is continuous in (x, λ). Hence

LEMMA I. *If throughout a certain range of values of* λ *the coefficients of* L *are continuous functions of* (x, λ) *and the coefficients of* $U_1, \cdots, U_n$ *are continuous functions of* λ, *and if for all values of* λ *in this range the system* (1) *has exactly* k*-fold compatibility; then if* $u_0(x)$ *denotes any particular solution of the system* (1) *when* $\lambda = \lambda_0$, *there exists a function* $u(x, \lambda)$ *continuous in* (x, λ) *which, throughout a certain neighborhood* of* λ_0, *satisfies* (1), *and is such that its limit for* $\lambda = \lambda_0$ *is* $u_0(x)$, *this limit being approached uniformly in* ab.

We turn next to

LEMMA II. *If* v *is any solution of the system*

$$M(v) = 0, \qquad V_i(v) = 0 \quad (i = 1, 2, \cdots, n) \tag{4}$$

adjoint† *to* (1), *and* u_g *is any solution of the system*

* This will be a one-sided neighborhood if λ_0 is an extremity of the range in question.

† For a definition of the adjoint system cf. for instance the paper already cited, where a more detailed statement of Green's theorem will also be found.

(5) $$L(u) = gu, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n),$$

then

(6) $$\int_a^b g u_\sigma v \, dx = 0.$$

The proof consists in applying Green's theorem

$$\int_a^b [vL(u) - uM(v)]dx = \sum_{i=1}^{2n} U_i(u)V_{2n+1-i}(v)$$

to the two functions u_σ and v, when it reduces at once to (6).

LEMMA III. *If the system* (1) *has k-fold compatibility* ($k \geqq 1$), *and* ϵ *is an arbitrarily given positive constant, a continuous, real function* $g(x)$ *exists such that* $0 \leqq g(x) < \epsilon$, *and that the system* (5) *has less than k-fold compatibility.*

To prove this, let u be a non-identically vanishing solution of (1) and v a similar solution of (4), which surely exists since (1) and (4) always have the same order of compatibility.* Since, by a fundamental (though seldom explicitly stated) theorem concerning homogeneous linear differential equations, neither u nor v has more than a finite number of zeros in ab, we can select a point p at which the product uv does not vanish. Either the real or the pure imaginary part of uv does not vanish at p; and without loss of generality we may assume that the former is the case as otherwise we might have multiplied v by a pure imaginary constant before beginning. Since uv, and therefore its real part, is a continuous function of x, we can surround p by an interval $a'b'$ so short that the real part of uv does not vanish there. Now define φ as a real continuous function of x which vanishes everywhere outside of $a'b'$ and is positive but less than ϵ everywhere within. We see then that

(7) $$\int_a^b \varphi u v \, dx \neq 0.$$

We now define the function g, whose existence is asserted in our lemma, by the equation

$$g = \lambda\varphi,$$

where λ is an, as yet undetermined, positive constant less than 1; and we see from (6) that

* Loc. cit., Theorem I.

$$\int_a^b \varphi u_g v\,dx = 0, \tag{8}$$

where u_g is any solution of (5).

Now assume Lemma III to be false. Then for all positive values of λ less than 1 the system (5) would have at least k-fold compatibility, while by Theorem I it cannot have more than k-fold compatibility for sufficiently small values of λ. Let us then restrict λ to values so small that (5) has always exactly k-fold compatibility. Then, by Lemma I, we can take for u_g a continuous function of (x, λ) which approaches $u(x)$ uniformly as λ approaches zero through positive values. Consequently

$$\lim_{\lambda=+0} \int_a^b \varphi u_g v\,dx = \int_a^b \varphi u v\,dx.$$

This, however, is in contradiction with formulas (7) and (8). Thus our lemma is proved.

We have indeed proved more than is stated in the lemma, for we have shown that g may be taken as identically zero except in the interval $a'b'$, which interval could be taken as short as we please and in any position we please provided it avoids a finite number of points. If now the order of compatibility of (5) is not zero, we can start afresh with this system, in place of (1), and, applying Lemma III to it, form a new system

$$L(u) = gu + g_1 u, \qquad U_i(u) = 0 \quad (i = 1, 2, \cdots, n)$$

which has a still lower order of compatibility and where $0 \leqq g_1 < \epsilon$. Moreover g_1 can be made to vanish everywhere except in an interval $a''b''$ as short as we please and not overlapping the interval $a'b'$. Hence the function $g + g_1$ satisfies the same inequality as g and g_1. Proceeding in this way step by step, we finally come to a system which is incompatible. Since all the intervals $a'b'$, $a''b''$, etc., which we use may be taken, if we wish, within an arbitrarily chosen subinterval of ab, we may state our final result as follows:

THEOREM II. *If ϵ is an arbitrarily given positive constant, a continuous, real function $g(x)$ exists such that $0 \leqq g(x) < \epsilon$ and such that the system* (5) *is incompatible. This function g may be taken to be identically zero except in an arbitrarily chosen subinterval of ab.*

We have proved this theorem, it is true, only when $k > 0$. If $k = 0$ it is, however, merely an obvious consequence of Theorem I.

We come now at last to our most important result, though one which is, at bottom, less far reaching than Theorem II, namely

THEOREM III. *If ϵ is an arbitrarily given positive constant, a continuous, real function $g(x)$ exists such that $0 < g(x) < \epsilon$ and such that the system* (5) *is incompatible.*

The proof consists simply in noticing that if we add to the function $g(x)$ determined in Theorem II a sufficiently small function everywhere positive (not zero), the system (5) will, by Theorem I, remain incompatible.*

This theorem is useful in making connection, by the method originally given in special cases by Hilbert, between the system (1) and an integral equation of the second kind.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

THE SMALLEST CHARACTERISTIC NUMBERS IN A CERTAIN EXCEPTIONAL CASE.

BY PROFESSOR MAXIME BÔCHER

(Read before the American Mathematical Society, September 8, 1914.)

THE characteristic numbers of the system

$$\frac{d}{dx}(ku') + (\lambda g - l)u = 0, \quad (k > 0,\ l \geqq 0), \tag{1}$$

$$\alpha u'(a) - \alpha' u(a) = 0, \quad (\alpha\alpha' \geqq 0,\ |\alpha| + |\alpha'| > 0), \tag{2}$$

$$\beta u'(b) + \beta' u(b) = 0, \quad (\beta\beta' \geqq 0,\ |\beta| + |\beta'| > 0) \tag{3}$$

are those values of λ for which (1) has a solution not identically zero which satisfies (2) and (3). We assume that k, g, l are continuous real functions of x in the interval $a \leqq x \leqq b$,

* A similar method enables us to deduce from Theorem II a great variety of other results, for instance:

If ϵ is an arbitrarily given positive constant, and $x_1, \cdots, x_p$ are arbitrarily given points in ab, there exists a continuous, real function $g(x)$ which vanishes and changes sign at each of the points x_i but vanishes nowhere else in ab, which satisfies the condition $|g(x)| < \epsilon$, and for which (5) *is incompatible.*

and that α, α', β, β' are real constants. It is with the real characteristic values we shall be concerned; indeed it is easily proved that there are no others.

If $g \geqq 0$, but not identically zero, we have obviously a special case of the system considered by Sturm in his first memoir of 1836, and indeed one which Sturm, and later Liouville, considered explicitly. If g changes sign, we have a case which has been supposed to be essentially different from Sturm's, and has been several times treated on this basis by more or less independent methods. I believe I was the first to point out* that if we divide (1) by $|\lambda|$ the system becomes merely a special case of the one originally treated by Sturm, whose results tell us that the characteristic numbers consist of two infinite systems, $\lambda_0^+, \lambda_1^+, \cdots$ and $\lambda_0^-, \lambda_1^-, \cdots$, the first set positive, the second negative (we suppose these sets to be both arranged according to increasing absolute values), such that the characteristic function corresponding to any one of them has in the interval $a < x < b$ a number of zeros exactly equal to the subscript of the λ in question.

There is, however, one exceptional case in which the result just stated becomes accurate only after a slight modification, namely that in which

$$l \equiv 0, \qquad \alpha' = \beta' = 0.$$

In this case, and in this case only, $\lambda = 0$ is a characteristic number, a corresponding characteristic function being $u = 1$. In this case also Sturm's results assert the existence of the characteristic numbers λ_i^+ and λ_i^-, when $i \geqq 1$, and the fact that the corresponding characteristic functions vanish just i times. It is not, however, immediately obvious whether the numbers λ_0^+ and λ_0^- still exist. As a matter of fact at least one, and sometimes both of them, will have disappeared, or, if we prefer, will reduce to the value $\lambda = 0$. The facts in this special case were established by Picone† by a different method. I wish here to show how very readily they follow by means of Sturm's results.

* In a lecture delivered at the 5th International Congress of Mathematicians at Cambridge, England, 1912. See Proceedings, vol. 1, pp. 173, 176. The literature there cited should have included a reference to Hilb, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 16 (1907), p. 279, where a sketch for a treatment of a special case in which g changes sign is given, viz. $l = 0$, $\alpha = \beta = 0$.

† *Annali della R. Scuola Normale di Pisa*, vol. 11 (1909), p. 39.

Let $v(x, \lambda)$ be the solution of (1) for which $v(a, \lambda) = 1$, $v'(a, \lambda) = 0$. According to Sturm's second theorem of comparison,* as λ increases from the value 0, the function

$$\frac{k(b)v'(b, \lambda)}{\lambda v(b, \lambda)} \tag{4}$$

decreases until it becomes negatively infinite. This occurs when λ reaches the value, μ_1, for which $v(b, \mu_1) = 0$. The characteristic number λ_0^+, if it exists, must evidently lie in the interval $0 < \lambda < \mu_1$, since a characteristic function corresponding to a characteristic number greater than μ_1 would necessarily have at least one zero in ab.

Now (4) approaches a definite finite limit as λ approaches zero, namely

$$D = k(b)\left[\frac{\partial}{\partial\lambda}v'(b, \lambda)\right]_{\lambda=0}. \tag{5}$$

Consequently, if $D > 0$, the quantity (4) in its decrease starts from positive values and passes, for some value of λ less than μ_1, through the value zero. This is the characteristic value λ_0^+, since for it $v'(b, \lambda) = 0$. On the other hand, if $D \leqq 0$, (4) does not pass through the value zero, and λ_0^+ does not exist.

In just the same way, by considering negative values of λ and dividing (1) by $|\lambda|$, we infer from Sturm's second theorem of comparison that, if $D < 0$, λ_0^- exists, if $D \geqq 0$, it does not.

It remains merely to evaluate D in terms of given quantities. From (1) we have

$$\frac{\partial}{\partial x}[k(x)v'(x, \lambda)] = -\lambda g(x)v(x, \lambda).$$

Integrating, and remembering that $v'(a, \lambda) = 0$, we get

$$k(b)v'(b, \lambda) = -\lambda\int_a^b g(x)v(x, \lambda)dx,$$

or, on differentiating with regard to λ,

$$k(b)\frac{\partial}{\partial\lambda}v'(b, \lambda) = -\int_a^b g(x)v(x, \lambda)dx - \lambda\int_a^b g(x)\frac{\partial}{\partial\lambda}v(x, \lambda)dx.$$

If here we let $\lambda = 0$, we get the desired formula

$$D = -\int_a^b g(x)dx. \tag{6}$$

* See, for instance, *Transactions*, vol. 1 (1900), p. 420.

All the steps we have here taken are justified since $v(x, \lambda)$ and $v'(x, \lambda)$ are both continuous in (x, λ) and analytic in λ.*

We have thus established Picone's result:

The differential system

$$\frac{d}{dx}(ku') + \lambda gu = 0, \qquad (k > 0)$$

$$u'(a) = 0, \qquad u'(b) = 0$$

in which g changes sign in ab has, if $\int_a^b g\,dx = 0$, *no characteristic number other than zero for which the characteristic function does not vanish, otherwise it has just one such characteristic number, namely a positive one if* $\int_a^b g\,dx < 0$, *a negative one if* $\int_a^b g\,dx > 0$.

I note in closing that the case $\int_a^b g\,dx = 0$ is of interest as giving one of the simplest examples of a characteristic number ($\lambda = 0$) whose order of multiplicity when regarded as a root of the characteristic equation (2 in this case) is not equal to its index (1 in this case), i. e., the number of linearly independent characteristic functions corresponding to it.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
July 15, 1914.

ON APPROXIMATION BY TRIGONOMETRIC SUMS.

BY PROFESSOR T. H. GRONWALL.

(Read before the American Mathematical Society, December 31, 1913.)

IN his paper "On approximation by trigonometric sums and polynomials"† Dr. Jackson has shown that, $f(x)$ being a function of period 2π and satisfying the Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$$

* For the fundamental properties of this important class of functions see my paper "On semi-analytic functions of two variables," *Annals of Mathematics*, 2d ser., vol. 12 (1910), p. 18. I was not aware when I published this article that some of these properties had been already given by Dini, *Annali di Matematica*, ser. 3, vol. 12 (1906), p. 179.

† *Transactions*, vol. 13 (1912), pp. 491–515.

for all values of x_2 and x_1, then there exists, for every integer n, a trigonometric sum of order not exceeding n

$$T_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx,$$

approximating $f(x)$ in such a way that for all values of x

$$|f(x) - T_n(x)| < \frac{4J'_m}{J_m} \cdot \frac{\lambda}{n},$$

where the integer m is determined by the condition $2m - 2 \leqq n < 2m$, and

$$(1) \qquad J_m = \frac{1}{m^3}\int_0^{\pi/2} \left(\frac{\sin mu}{\sin u}\right)^4 du, \qquad J'_m = \frac{1}{m^2}\int_0^{\pi/2} u\left(\frac{\sin mu}{\sin u}\right)^4 du.$$

By asymptotic considerations, Dr. Jackson shows that for $m \geqq 4$, $n \geqq 6$,

$$(2) \qquad \frac{4J'_m}{J_m} \leqq 2.90\cdots$$

It is the purpose of the present note to show that the quotient in (2) decreases as m increases,

$$(3) \qquad \frac{J'_{m+1}}{J_{m+1}} - \frac{J'_m}{J_m} < 0 \qquad (m = 1, 2, 3, \cdots).$$

Since it may be shown that*

$$(4) \qquad J_m = \frac{\pi}{3}\left(1 + \frac{1}{2m^2}\right),$$

the inequality (3) is equivalent to

$$(5) \qquad \Delta_m = (2m^2 + 1)\int_0^{\pi/2} u \frac{\sin^4 (m+1)u - \sin^4 mu}{\sin^4 u}\, du - (4m + 3)\int_0^{\pi/2} u\left(\frac{\sin mu}{\sin u}\right)^4 du < 0.$$

* T. H. Gronwall, "On the degree of convergence of Laplace's series," *Transactions*, vol. 15 (1914), pp. 1–30.

We shall begin by replacing u by an approximate expression in terms of $\sin u$. We have

$$\frac{d}{du}\frac{u-\sin u}{\sin^3 u} = \frac{\sin u - 3u\cos u + 2\sin u\cos u}{\sin^4 u}$$

and

$$\frac{d}{du}(\sin u - 3u\cos u + 2\sin u\cos u) = 3\sin u(u-\sin u) + (1-\cos u)^2 > 0 \text{ for } 0 < u < \pi/2;$$

therefore $\sin u - 3u\cos u + 2\sin u\cos u > 0$ for $0 < u < \pi/2$, and consequently $\frac{u-\sin u}{\sin^3 u}$ increases monotonely with u for $0 < u < \pi/2$, so that

$$u = \sin u + \eta(u)\sin^3 u,$$

$$(6)\qquad \frac{1}{6} = \eta(0) < \eta(u) < \eta\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 \text{ for } 0 < u < \frac{\pi}{2}.$$

The expression (5) for Δ_m now gives

$$\begin{aligned}\Delta_m = (2m^2+1)\int_0^{\pi/2}\frac{\sin^4(m+1)u - \sin^4 mu}{\sin^3 u}\,du \\ - (4m+3)\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du \\ + (2m^2+1)\int_0^{\pi/2}\eta(u)\frac{\sin^4(m+1)u - \sin^4 mu}{\sin u}\,du \\ - (4m+3)\int_0^{\pi/2}\eta(u)\frac{\sin^4 mu}{\sin u}\,du \\ (7)\qquad < (2m^2+1)\int_0^{\pi/2}\frac{\sin^4(m+1)u - \sin^4 mu}{\sin^3 u}\,du \\ - (4m+3)\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du \\ + (2m^2+1)\left(\frac{\pi}{2}-1\right)\int_0^{\pi/2}\frac{\sin^4(m+1)u - \sin^4 mu}{\sin u}\,du \\ - \frac{4m+3}{6}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du.\end{aligned}$$

Proceeding to evaluate our various integrals, we find, integrating by parts,

$$\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du = \int_0^{\pi/2} \frac{\sin^4 mu}{\sin u} d(-\cot u)$$

$$= \int_0^{\pi/2} \cot u \frac{d}{du}\left(\frac{\sin^4 mu}{\sin u}\right) du$$

$$= \int_0^{\pi/2} \frac{4m \sin^3 mu \cos mu \cos u}{\sin^2 u}\,du$$

$$- \int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du + \int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du,$$

or

$$\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du = 2m \int_0^{\pi/2} \sin^3 mu \cos mu\, d\left(-\frac{1}{\sin u}\right)$$

$$+ \frac{1}{2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du$$

$$= 2m \int_0^{\pi/2} \frac{1}{\sin u} \frac{d}{du}(\sin^3 mu \cos mu)du$$

$$+ \frac{1}{2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du$$

$$= 2m^2 \int_0^{\pi/2} \frac{\sin^2 2mu - \sin^2 mu}{\sin u}\,du$$

$$+ \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2 mu - \frac{1}{4}\sin^2 2mu}{\sin u}\,du.$$

Now the identity

$$\frac{\sin^2 nu}{\sin u} = \sum_{\lambda=0}^{n-1} \sin (2\lambda + 1)u$$

gives

$$\int_0^{\pi/2} \frac{\sin^2 nu}{\sin u}\,du = \sum_{\lambda=0}^{n-1} \frac{1}{2\lambda + 1},$$

and consequently

$$\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^2 u}\,du = 2m^2 \sum_{\lambda=m}^{2m-1} \frac{1}{2\lambda+1} + \frac{1}{2}\sum_{\lambda=0}^{m-1} \frac{1}{2\lambda+1} - \frac{1}{8}\sum_{\lambda=0}^{2m-1} \frac{1}{2\lambda+1},$$

(8)

$$\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du = \sum_{\lambda=0}^{m-1} \frac{1}{2\lambda+1} - \frac{1}{4}\sum_{\lambda=0}^{2m-1} \frac{1}{2\lambda+1}.$$

From these equations and the last part of (7) we obtain

$$\Delta_m < -2(m^2-2m-1)\sum_{\lambda=m}^{2m-1} \frac{1}{2\lambda+1} - \frac{2}{3}(4m+3)\left(\sum_{\lambda=0}^{m-1} \frac{1}{2\lambda+1} - \frac{1}{4}\sum_{\lambda=0}^{2m-1} \frac{1}{2\lambda+1}\right) + (2m^2+1)\left(\frac{\frac{\pi}{2}-1}{2m+1} + \frac{1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} - \frac{1}{4}\cdot\frac{1}{4m+3}\right).$$

For $m > 2$, the first term in the expression on the right is negative, and since the difference of the two sums in the second term obviously increases with m, we have for $m > 2$

$$\sum_{\lambda=0}^{m-1} \frac{1}{2\lambda+1} - \frac{1}{4}\sum_{\lambda=0}^{2m-1} \frac{1}{2\lambda+1} > 1 + \tfrac{1}{3} - \tfrac{1}{4}(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}) = \tfrac{23}{35};$$

furthermore

$$\frac{\pi-2}{4m+2} + \frac{1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} - \frac{1}{4}\cdot\frac{1}{4m+3} < \frac{\pi-2+1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} < \frac{2}{4m+1},$$

so that finally, for $m > 2$,

$$\Delta_m < -\tfrac{2}{3}\cdot\tfrac{23}{35}(4m+3) + \frac{4m^2+2}{4m+1} < 0,$$

which proves our theorem for $m > 2$. For $m = 1$ and $m = 2$ it is readily verified from the numerical values of J_m and J'_m given by Jackson.

From (1), (6) and (8) it follows that

$$J'_m = \frac{1}{m^2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du + \eta' \cdot \frac{1}{m^2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du,$$

$$\tfrac{1}{6} < \eta' < \left(\frac{\pi}{2} - 1\right),$$

and consequently

$$J'_m = 2\sum_{\lambda=m}^{2m-1} \frac{1}{2\lambda+1} + O\left(\frac{\log m}{m^2}\right),$$

$$\lim_{m=\infty} J'_m = \lim_{m=\infty} \sum_{\lambda=1}^{2m-1} \frac{2}{2\lambda+1} = \int_0^1 \frac{dx}{1+x} = \log 2.$$

Using this result in connection with (4), it is seen that

$$\lim_{m=\infty} \frac{4J'_m}{J_m} = \frac{12 \log 2}{\pi} = 2.648 - ,$$

so that, using the numerical values of J_m and J'_m we may finally state the result

$$2.758 - = \frac{4J_4'}{J_4} > \frac{4J_5'}{J_5} > \frac{4J_6'}{J_6} > \cdots > 2.648 - .$$

PRINCETON UNIVERSITY,
December 20, 1913.

NOTE ON THE ROOTS OF ALGEBRAIC EQUATIONS.

BY PROFESSOR R. D. CARMICHAEL AND DR. T. E. MASON.

(Read before the American Mathematical Society at Chicago, April 10, 1914.)

1. LANDAU* has established certain interesting inequalities concerning the least root of a class of algebraic equations, having been led to these results by considerations connected with his remarkable extension and generalization of Picard's famous theorem to the effect that an entire function which fails to assume two values is a constant. These special in-

* *Annales de l'École Normale Supérieure* (3), vol. 24 (1907), pp. 179–201; *Vierteljahrsschrift der Naturf. Gesellschaft in Zürich*, vol. 51 (1906), pp. 252–318.

equalities have been generalized and made more precise by Allardice.* Finally, still more comprehensive results have been obtained by Fejér† by a method which leaves nothing to be desired in the way of simplicity and elegance. Fejér's result, which contains those of the other authors mentioned, may be stated as follows:

Let

$$a_0 + a_1 z^{\nu_1} + a_2 z^{\nu_2} + \cdots + a_k z^{\nu_k} = 0 \tag{1}$$

be an equation of $k+1$ *terms in which*

$$a_0 \neq 0,\ a_1 \neq 0,\ \nu_1 < \nu_2 < \cdots < \nu_k.$$

Let ζ *be a root of this equation of least absolute value. Then we have the inequalities*

$$|\zeta| \leqq \left[\frac{\nu_2\,\nu_3 \cdots \nu_k}{(\nu_2-\nu_1)(\nu_3-\nu_1)\cdots(\nu_k-\nu_1)}\right]^{1/\nu_1} \left|\frac{a_0}{a_1}\right|^{1/\nu_1}, \tag{2}$$

$$|\zeta| \leqq \left[\frac{(\nu_1+1)(\nu_1+2)\cdots(\nu_1+k-1)}{1\cdot 2\cdots(k-1)}\right]^{1/\nu_1} \left|\frac{a_0}{a_1}\right|^{1/\nu_1}. \tag{3}$$

In case $\nu_1 = 1$, *we have*

$$|\zeta| \leqq k\left|\frac{a_0}{a_1}\right|; \tag{4}$$

and this value is actually attained by the roots of the equation

$$a_0\left(1+\frac{a_1 z}{k a_0}\right)^k = 0.$$

For the case $\nu_1 = 1$ formula (2) was obtained by Allardice. For the case when $k=2$ and $\nu_1 = 1$ formula (4) was established by Landau. Furthermore, for $k=3$ and $\nu_1 = 1$ Landau proved that $|\zeta|$ is not greater than $5\frac{2}{3}a_0/a_1$, a result less precise than that of Fejér above.

The principal purpose of this note is to establish other results concerning the least roots of algebraic equations, especially of those of certain special forms. It will be seen that these results are in some cases more far-reaching than those of Fejér. An additional theorem of a related nature is also given in the final section.

* BULLETIN, vol. 13 (1907), pp. 443–447.
† *Comptes rendus*, vol. 145 (1907), pp. 459–461.

It is convenient, first of all, to simplify equation (1) in the following way: Divide each member by a_0 and replace z by x, where

$$x = \left(\frac{a_1}{a_0}\right)^{1/\nu_1} z.$$

Then the equation takes the form

$$1 + x^{\nu_1} + \alpha_2 x^{\nu_2} + \alpha_3 x^{\nu_3} + \cdots + \alpha_k x^{\nu_k} = 0. \tag{5}$$

We shall confine our attention principally to the latter equation.

2. We shall consider first the case in which $\nu_1 = 1$ and $\nu_{i+1} = \nu_i + 1$ when $i > 1$, writing the equation in the form

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_m x^m = 0 \quad (s \geqq 1). \tag{6}$$

Let $x_1, x_2, \cdots, x_m$ denote the roots of this equation. From the equation whose roots are the reciprocals of those of (6) we see at once that

$$\Sigma \frac{1}{x_1} = -1, \ \Sigma \frac{1}{x_1 x_2} = 0, \ \cdots, \ \Sigma \frac{1}{x_1 x_2 \cdots x_s} = 0,$$

where the summation in each case is for all terms of the type written. From these equations and the customary formulas for the sum of the roots of an equation in terms of the elementary symmetric functions of these roots* it follows readily that

$$\Sigma \frac{1}{x_1^s} = (-1)^s.$$

Therefore

$$\frac{1}{|x_1|^s} + \frac{1}{|x_2|^s} + \cdots + \frac{1}{|x_m|^s} \geqq 1.$$

Now if we suppose that x_1 is a root of (6) of least absolute value, we have

$$\frac{m}{|x_1|^s} \geqq 1;$$

whence

$$|x_1| \leqq \sqrt[s]{m}.$$

* See Bôcher's Higher Algebra, p. 244.

This result may be stated in the form of the following theorem:

I. *The equation*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_m x^m = 0 \quad (s \geqq 1),$$

has a root which is not greater in absolute value than $\sqrt[s]{m}$, *whatever values* $c_{s+1}, c_{s+2}, \cdots, c_m$ *may have.*

COROLLARY. *The equation*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_{2^s}x^{2^s} = 0 \quad (s \geqq 1), \tag{7}$$

has a root which is not greater than 2 in absolute value.

The result of this corollary for the special case when $c_{s+2} = c_{s+3} = \cdots = c_{2^s} = 0$ is given by Landau (loc. cit.). Here the equation reduces to a trinomial.

If we apply to equation (7) the Allardice-Fejér formula (2) for $\nu_1 = 1$ we conclude that this equation has a root not greater in absolute value than $2^s/s$. For s approaching infinity this value $2^s/s$ itself approaches infinity at a rapid rate. Hence, the circle in which, according to the Allardice-Fejér result, equation (7) certainly has a root increases in size with increasing s. But from the theorem of our corollary it is seen that (7) always has a root in the circle of radius 2 about the point zero. Thus for the special case of equation (7) (and likewise of (6), as one may readily show) our results reach further than those of Allardice and Fejér.

We shall now show inversely that there is in general but one root of equation (6) which is bounded by the circle about zero of radius $\sqrt[s]{m}$ when $c_{s+1}, \cdots, c_m$ are arbitrary. More precisely, we shall prove the following theorem:

II. *Let* s *and* m *be any two positive integers having a prime number* p *between them:* $s < p < m$. *Let* M *be any positive constant. Then there exist equations of the form*

$$1 + x + c_{s+1}x^{s+1} + \cdots + c_m x^m = 0$$

having every root but one greater than M *in absolute value.*

To prove this theorem it is sufficient to construct equations having the specified property. Let kp($k \geqq 1$) be the greatest multiple of p which is less than m. We shall further specialize the equation to be constructed so that it shall have the form

$$1 + x + c_p x^p + c_{p+1}x^{p+1} + \cdots + c_{kp+1}x^{kp+1} = 0. \tag{8}$$

Now let ω be any primitive pth root of unity; then the set of pth roots of unity is $1, \omega, \omega^2, \cdots, \omega^{p-1}$. Let $x_1, x_2, \cdots, x_{kp+1}$ denote the $kp+1$ roots of equation (8). We shall build up equation (8) by properly choosing the values of its roots. Thus, we put

$$x_{kp+1} = -1,$$

$$x_{\mu k+\nu} = \omega^{-\mu}\tau, \quad (\mu = 0, 1, \cdots, p-1;\ \nu = 1, 2, \cdots, k),$$

where τ is any number whatever which is different from zero.

With these values of $x_1, x_2, \cdots, x_{kp+1}$ we have

$$\Sigma \frac{1}{x_1^t} = (-1)^t \quad (t = 1, 2, \cdots, p-1),$$

as one sees readily by means of the well-known relations

$$1^t + \omega^t + \omega^{2t} + \cdots + \omega^{(p-1)t} = 0 \quad (t = 1, 2, \cdots, p-1).$$

From the usual formulas (Bôcher, loc. cit.) for the sum of the roots of an equation in terms of the elementary symmetric functions of these roots it is now easy to see that

$$\Sigma \frac{1}{x_1} = -1, \quad \Sigma \frac{1}{x_1 x_2} = 0, \quad \Sigma \frac{1}{x_1 x_2 x_3} = 0, \cdots, \quad \Sigma \frac{1}{x_1 x_2 \cdots x_{p-1}} = 0.$$

Hence, the equation of which the roots are $x_1, x_2, \cdots, x_{kp+1}$ has the form (8). If $|\tau| > M$, then all but one of the roots of the equation so formed are greater than M in absolute value. Hence we conclude to the theorem as stated above.

By means of II and the corollary to I we shall now prove the following theorem:

III. *Every equation of the form*

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_n x^n = 0 \quad (s \geqq 1),$$

has at least one root not greater than 2 in absolute value, while special equations of this form may have all roots but one greater in absolute value than any preassigned M.

In order to complete the demonstration of this result it is sufficient to observe that obviously only one root is bounded when $s = 1$ and that a prime number p lies between s and 2^s when $s \geqq 2$. For $s = 2$ we have $p = 3$ and for $s = 3$ we have $p = 5$ or 7. For $s > 3$ we may prove the existence of p by means of Tschebyschef's theorem*: If an integer s is greater

* Bachmann, Niedere Zahlentheorie I, p. 66.

than 3 then there is at least one prime number between s and $2s - 2$.

3. Next, let us consider the equation

$$1 + x^r + c_{sr+1}x^{sr+1} + \cdots + c_m x^m = 0 \qquad (s \geqq 1). \tag{9}$$

Let $x_1, x_2, \cdots, x_m$ be the roots of this equation and denote by A_k and S_k respectively, the sums

$$A_k = \Sigma \frac{1}{x_1 x_2 \cdots x_k}, \quad S_k = \Sigma \frac{1}{x_1{}^k}.$$

Then we have*

$$S_m - A_1 S_{m-1} + A_2 S_{m-2} - \cdots + (-1)^{m-1} A_{m-1} S_1 + (-1)^m m A_m = 0. \tag{10}$$

Now from equation (9) we see readily that

$$A_1 = A_2 = \cdots = A_{r-1} = 0, \ A_r = (-1)^r, \ A_{r+1} = \cdots = A_{sr} = 0.$$

Hence it follows from (10) that

$$S_r = -r;$$

and then that

$$S_{2r} = r, \text{ if } s \geqq 2.$$

Continuing thus we have finally

$$S_{sr} = (-1)^s r;$$

that is,

$$\frac{1}{x_1{}^{sr}} + \frac{1}{x_2{}^{sr}} + \cdots + \frac{1}{x_m{}^{sr}} = (-1)^s r.$$

If x_1 is a root of (9) of least absolute value, then from the last equation it follows readily that

$$\frac{m}{|x_1|^{sr}} \geqq r \quad \text{or} \quad |x_1| \leqq \sqrt[sr]{\frac{m}{r}}.$$

Thus we have the following generalization of theorem I:

IV. *Every equation of the form* (9) *has at least one root which is less in absolute value than* $\sqrt[sr]{m/r}$.

* See Bôcher's Higher Algebra, p. 244.

We may also generalize theorem II and so obtain the following result:

V. *Let r, s and m be any three positive integers such that r is prime while a prime number p lies between sr and $m-r+1$; $sr<p<m-r+1$. Let M be any positive constant. Then there exist equations of the form* (9) *which have all but r of their roots greater than M in absolute value.*

The detailed proof of this theorem will not be given. It is sufficient to construct an equation of the form

$$1+x^r+c_px^p+c_{p+1}x^{p+1}+\cdots+c_{kp+r}x^{kp+r}=0 \quad (kp+r\leqq m),$$

by means of its roots $x_1, x_2, \cdots, x_{kp+r}$ defined as follows:

$$x_{\mu k+\nu}=\omega^{-\mu}\tau, \ (\mu=0, 1, 2, \cdots, p-1; \ \nu=1, 2, \cdots, k);$$

$$x_{kp+t}=\eta\epsilon^{-t} \quad (t=1, 2, \cdots, r),$$

ω being a primitive pth root of unity, ϵ a primitive rth root of unity and η any rth root of -1.

4. Related to the foregoing theorems is the following rather obvious one:

VI. *In the equation*

$$1+a_1x+\cdots+a_kx^k+\cdots+a_mx^m=0, \tag{11}$$

of degree m, let a_k be any given non-zero coefficient and let all the other coefficients be chosen in any manner whatever. The equation has at least one root which is not greater in absolute value than

$$\sqrt[k]{\frac{m!}{k!(m-k)!}}\cdot\sqrt[k]{\frac{1}{|a_k|}}. \tag{12}$$

The equation

$$\left(1+x\sqrt[k]{\frac{k!(m-k)!a_k}{m!}}\right)^m=1+a_1x+\cdots+a_kx^k+\cdots+a_mx^m=0$$

has all its roots equal in absolute value to the quantity (12).

Let $x_1, x_2, \cdots, x_m$ denote the roots of the given equation and let x_1 be a root of least absolute value. We have readily

$$\Sigma\frac{1}{x_1x_2\cdots x_k}=(-1)^ka_k; \tag{13}$$

whence

$$\frac{1}{|x_1|^k}\frac{m!}{k!(m-k)!} \geqq |a_k|,$$

since the number of terms in the first member of (13) is obviously equal to the number of combinations of m things taken k at a time. Hence,

$$|x_1| \leqq \sqrt[k]{\frac{m!}{k!(m-k)!}}\sqrt[k]{\frac{1}{|a_k|}}.$$

This proves the first part of the theorem. The second part is obviously true.

5. We shall now prove a theorem of a nature somewhat different from that of those in the preceding sections; namely, the following:

VII. *All the roots of the equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{14}$$

are in absolute value less than or equal to

$$\sqrt{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}.$$

Let ζ be a root of least absolute value of the equation

$$1 + a_1z + a_2z^2 + \cdots + a_nz^n = 0.$$

Write

$$\frac{1}{1 + a_1z + a_2z^2 + \cdots + a_nz^n} = 1 + c_1z + c_2z^2 + \cdots. \tag{15}$$

Now, the circle of convergence of the power series in the second member of (15) passes through the point ζ. Its radius is therefore equal to $|\zeta|$. Therefore, by a well-known property of power series, we have

$$\frac{1}{|\zeta|} = \limsup_{m=\infty} \sqrt[m]{c_m}.$$

Hence, if $\bar{x}$ is a root of (14) of greatest absolute value, it is clear that

$$|x| = \limsup_{m=\infty} \sqrt[m]{c_m}. \tag{16}$$

Now, the value of c_m is readily expressed in the form of a determinant, as follows: If we multiply equation (15) through

by the denominator of its first member and in the result equate coefficients of like powers of z, we have

$$\begin{aligned} c_1 &= -a_1, \\ a_1c_1 + c_2 &= -a_2, \\ a_2c_1 + a_1c_2 + c_3 &= -a_3, \\ a_3c_1 + a_2c_2 + a_1c_3 + c_4 &= -a_4, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \end{aligned}$$

Hence

$$c_m = -\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & a_1 \\ a_1 & 1 & 0 & 0 & \cdots & 0 & a_2 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 & a_3 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & a_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{17}$$

Now, if Δ_m is a determinant of order m,

$$\Delta_m = \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mm} \end{vmatrix},$$

then

$$|\Delta_m| \leqq \sqrt{\sigma_1\sigma_2 \cdots \sigma_m},$$

where

$$\sigma_i = \sum_{j=1}^{m} |a_{ij}|^2.$$

This fundamental theorem is due to Hadamard.*

From this theorem and equation (17) we see readily that

$$|c_m| < \{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2\}^{m/2}, \tag{18}$$

since each row in the determinant in (17) has the property that the sum of the squares of the absolute values of its elements is not greater than $1 + |a_1|^2 + \cdots + |a_n|^2$. From (18) and the result associated with (16) the theorem now follows immediately.

INDIANA UNIVERSITY,
February, 1914.

* *Bull. des Sciences Math.* (Darboux), vol. 17 (1893), pp. 240–246.

REMARKS ON FUNCTIONAL EQUATIONS.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, September 8, 1913.)

1. THE remarks of this article are in continuation of our paper in the BULLETIN, volume 19 (1912), pages 66–70. We have mentioned that a class of functional equations arises from the solution of the equation

$$\frac{\partial f(x, y)}{\partial x} + \mu(x, y)\frac{\partial f(x, y)}{\partial y} = 0, \tag{1}$$

where in particular we have assumed that $\mu(x, y) = \psi'(x)/\psi'(y)$. It is obvious that other classes of functional relations are obtained by considering the solution of the equation

$$\frac{dy}{dx} = \mu(x, y)$$

in connection with (1) when $\mu(x, y)$ is suitably specialized. The equation (1) may profitably be compared with another source of functional equations, that is, the equation*

$$\frac{\partial \varphi(x, y)}{\partial x} + \frac{\partial \varphi(x, y)}{\partial y} = \lambda(x, y). \tag{2}$$

The following functional relation is derivable from (1) and admits an explicit solution:†

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\}. \tag{1'}$$

This equation can be solved in another manner by noticing that $\phi(x)$ can be formally eliminated. The result is

$$\Omega\{x - y\} = \Omega(x)\psi_1^{-1}(y) - \Omega(y)\psi_1^{-1}(x), \tag{1''}$$

where

$$\Omega(0) = 0, \quad \psi_1^{-1}(0) = 1,$$

$$\psi_1^{-1}(x) = \psi^{-1}\{x + \psi(1)\}.$$

* Cf. V. Volterra, *Acc. Lincei*, vol. 19 (1910), pp. 169, 425; BULLETIN, vol. 19, p. 171; G. C. Evans, Proc. Cong. Math., Cambridge, 1913, vol. 1, p. 387; J. Hadamard, *L'Enseignement Mathématique*, vol. 14 (1912), p. 18, note; Paul Lévy, Paris thesis.

† Cf. BULLETIN, l. c., page 67.

We can now proceed as indicated by Abel, Œuvres, volume 1, page 10.

2. We consider the relations*

$$f(x, y) = \chi_f^{-1}\{\chi_f(x) - \chi_f(y)\}. \tag{3}$$

$$\phi(x, y) = \chi_f^{-1}\{\chi_f(x) + \chi_f(y)\}. \tag{4}$$

In (3) and (4) we write $\chi_f^{-1}(x)$ and $\chi_f^{-1}(y)$ for x and y respectively; then

$$f[\chi_f^{-1}(x), \chi_f^{-1}(y)] = \chi_f^{-1}\{x - y\}. \tag{3'}$$

$$\phi[\chi_f^{-1}(x), \chi_f^{-1}(y)] = \chi_f^{-1}\{x + y\}. \tag{4'}$$

Thus (3') and (4') represent respectively a subtraction formula and an addition formula.† It is thus seen that under differentiation, etc., symmetric and quasi-transitive‡ functional properties give rise to addition and subtraction formulas. In the philosophy of mathematics this result is of considerable interest. In symmetry and transitiveness and formal properties closely allied to these, many mathematical disciplines are rooted, e. g., the foundations of geometry and the theory of quaternions. The fact that certain properties of the indicated category induce addition and subtraction formulas tends to corroborate the fundamental character of the latter in the theory of functions.

3. We have shown§ purely formally that if

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z, \tag{5}$$

then ϕ can be eliminated; we obtained

$$f\{y, f(y, z)\} = z. \tag{5'}$$

In a similar manner f can be eliminated; for we have

$$\phi\{f[\phi(z, y), z], z\} = \phi(z, y),$$

that is,

$$\phi(y, z) = \phi(z, y). \tag{5''}$$

The relation (5'') may be used in the proof of theorem 1,

* BULLETIN, l. c., p. 68.

† On this interpretation see Bourlet, *Ann. Sci. de l'Ecole Norm. Sup.*, vol. 14 (1897), pp. 140, 141. Independently of this reference, the preceding remark was kindly suggested to the author by Professor E. B. Van Vleck.

‡ BULLETIN, vol. 18 (1912), p. 300.

§ BULLETIN, vol. 19 (1912), p. 69.

BULLETIN, l. c., page 68, as expeditiously as relation (5′) in theorem 2, l. c., page 69. With regard to the duality, as it were, between the two theorems it is desirable to replace the hypothesis of theorem 1

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}$$

by the relation

$$\phi\{y, \phi(x, z)\} = \phi\{z, \phi(x, y)\}.$$

The correspondence between (5′) and (5″) is possibly of use elsewhere in analysis.

From the equations

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}, \tag{6}$$

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z,$$

we have shown l. c., page 69, that the ϕ may be formally eliminated; the result is

$$f\{f(x, z), f(x, y)\} = f(y, z). \tag{7}$$

We proceed to eliminate f from the equations (6). Substituting $\phi(x, z)$ for x in the first of equations (6) we get

$$f(y, x) = f\{z, f[\phi(x, z), y]\}.$$

Hence,

$$f\{\phi(y, z), x\} = f\{z, f[\phi(x, z), \phi(y, z)]\}.$$

Therefore*

$$f\{\phi[y, \phi(x, z)], x\} = \phi(y, z).$$

Substituting in this equation $f(x, z)$ for x we find

$$f\{\phi(y, x), f(x, z)\} = \phi(y, z). \tag{8}$$

Now

$$\phi\{f[\phi(y, x), f(x, z)], f(x, z)\} = \phi(y, x).$$

Therefore by (8),

$$\phi\{\phi(y, z), f(x, z)\} = \phi(y, x). \tag{9}$$

In (9) we write $\phi(z, x)$ for x; then

$$\phi\{\phi(y, z), x\} = \phi\{y, \phi(z, x)\}$$

or

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}. \tag{10}$$

*From (6) we have $f\{f(y, z), f(x, z)\} = f(y, x)$ and consequently $(x, y) = f\{\varphi(x, z), \varphi(y, z)\}$.

Hence Abel's functional equations* result by eliminating f from equations (6).

In a similar manner we may eliminate ϕ from the equations

$$\phi\{y, \phi(x, z)\} = \phi\{z, \phi(x, y)\}, \tag{11}$$

$$\phi\{f(y, z), z\} = y, \quad f\{\phi(y, z), y\} = z.$$

To do this, we easily derive from equations (11) the equation (9); thence using the relation

$$f\{\phi[f(z, y), \phi(x, y)], \quad f(z, y)\} = \phi(x, y)$$

we obtain equation (8). From (9) there results in connection with (11)

$$\phi\{f(x, y), z\} = \phi\{f(z, y), x\}. \tag{12}$$

And from (8) we derive

$$f\{z, f(y, x)\} = \phi\{f(z, y), x\}. \tag{13}$$

From (12) and (13) we get finally the desired relation

$$f\{x, f(y, z)\} = f\{z, f(y, x)\}. \tag{14}$$

The preceding formal eliminations enter into Grassmann's theory of synthetic and analytic associations which, as is well known, provides an abstract generalization of the processes of addition and subtraction.† In fact we may regard $\phi(x, y)$ and $f(x, y)$ as instances of Grassmann's associations $(x \cap y)$ and $(x \cup y)$ respectively; we have then in Grassmann's theory the theorems

I. $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

imply

$$(y \cap z) = (z \cap y) \text{ and } \{y \cup (y \cup z)\} = z.$$

II. $$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$$

and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

imply

$$\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}.$$

III. $$\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$$

and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$

* Œuvres, vol. 1, p. 61.

† Cf. Gesammelte Werke, vol. 1, part 1, pp. 33–45.

imply

$$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}.$$

IV. $$\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$$

$$\text{and } \{y \cup (y \cup z)\} = z$$

imply

$$\{(x \cup y) \cup (x \cup z)\} = (z \cup y).$$

The study of the preceding relationship naturally would lead one to consider the monograph of Pietzker, Beiträge zur Funktionenlehre, Leipzig, 1899.*

4. The reference to Grassmann† in the preceding section suggests the following problem:

Given the quasi-transitive function $f(x, y)$ to find the quasi-transitive function $f_1(x, y)$ such that

$$f_1\{f(x, y), z\} = f\{f_1(x, z), f_1(y, z)\}. \tag{15}$$

We have

$$f(x, y) = \chi^{-1}\{\chi(x) - \chi(y)\}, \tag{16}$$

$$f_1(x, y) = \chi_1^{-1}\{\chi_1(x) - \chi_1(y)\}.$$

In (15) we put $z = z_0$, a constant; then if $\psi(x) = f_1(x, z_0)$ we have

$$\psi\{f(x, y)\} = f\{\psi(x), \psi(y)\}. \tag{15'}$$

Substituting in this relation for $f(x, y)$ from (16) we obtain

$$\chi\psi\chi^{-1}(x - y) = \chi\psi\chi^{-1}(x) - \chi\psi\chi^{-1}(y). \tag{17}$$

Hence

$$\psi(x) = \chi^{-1}\{c\chi(x)\}, \tag{18}$$

where c is an arbitrary constant. On the other hand, we put in relation (16)

$$\chi_2(x) = c'\chi_1(x);$$

then

$$f_1(x, z) = \chi_2^{-1}\{\chi_2(x) - c'\chi_1(z)\} \tag{16'}$$

or for $z = z_0$

$$\psi(x) = \chi_2^{-1}\{\chi_2(x) - c''\}, \tag{19}$$

where c'' is an arbitrary constant. As is well known, func-

* Cf. Moritz, *Amer. Journ. of Mathematics*, vol. 24, No. 3; A. F. Carpenter, same journal, vol. 35, No. 1.

† L. c., pp. 41–42.

tional relations of the types (18) and (19) have been treated by Schroeder and Abel respectively.* From (18) and (19) we have

$$\chi^{-1}\{c\chi(x)\} = \chi_2^{-1}\{\chi_2(x) - c''\}$$

or

$$\chi_2\chi^{-1}(cx) = \chi_2\chi^{-1}(x) - c''$$

or

$$\chi_2\chi^{-1}(cx) = \chi_2\chi^{-1}(c) + \chi_2\chi^{-1}(x) - \chi_2\chi^{-1}(1). \tag{20}$$

The solution of (20) is

$$\chi_2\chi^{-1}(x) = \log_a c_1 x, \tag{21}$$

where c_1 is a constant and the base a is arbitrary. From (21) we have

$$\chi_2(x) = \log_a [c_1\chi(x)].$$

Hence

$$f_1(x, y) = \chi^{-1}\{c_1' \cdot \chi(x)/\chi(y)\} \tag{22}$$

which is the solution required.

Proceeding in a precisely analogous manner we can determine the symmetric function $\phi_1(x, y)$ such that

$$\phi_1\{\phi(x, y), z\} = \phi\{\phi_1(x, z), \phi_1(y, z)\},$$

where ϕ is a given symmetric function; we find

$$\phi_1(x, y) = \chi^{-1}\{c_2\chi(x) \cdot \chi(y)\}, \tag{23}$$

where c_2 is a constant and

$$\phi(x, y) = \chi^{-1}\{\chi(x) + \chi(y)\}.$$

A formal solution of a problem closely related to the above has been given by Pietzker, l. c., pages 9–10.

5. If we assume properties (6), (15) and the relation

$$f_1\{\phi_1(y, z), y\} = z, \tag{24}$$

then there exists a function $\chi(x)$ such that

$$\begin{aligned} \phi(x, y) &= \chi^{-1}\{\chi(x) + \chi(y)\}, \\ f(x, y) &= \chi^{-1}\{\chi(x) - \chi(y)\}, \\ f_1(x, y) &= \chi^{-1}\{\chi(x)/\chi(y)\}, \\ \phi_1(x, y) &= \chi^{-1}\{\chi(x) \cdot \chi(y)\}. \end{aligned} \tag{25}$$

* Cf. M. J. van Uven, *Kon. Akad. van Wetenschappen te Amsterdam*, vol. 12 (1909–1910), pp. 208, 427; A. Hurwitz, *Math. Annalen*, vol. 70 (1910–1911), p. 33.

Inasmuch as certain formal properties of the real number system are satisfied* by the preceding functions, it seems appropriate to call $\phi(x, y)$ the *pseudo-sum* of x and y; the function $\phi_1(x, y)$, the *pseudo-product* of x and y, and similarly for the functions $f(x, y)$ and $f_1(x, y)$. Moreover, to zero corresponds $f(x, x)$, a constant, and to unity corresponds $f_1(x, x)$, a constant.

We can, of course, further condition the functions $\phi(x, y)$, $f(x, y)$, etc., by referring to other properties of the real number system. For example, by assuming the relation

$$f(tx, ty) = tf(x, y) \tag{26}$$

in addition to the relation (7) we find that there exists a function $\chi(x)$ such that†

$$\chi\{t \cdot \chi^{-1}(x - y)\} = \chi\{t \cdot \chi^{-1}(x)\} - \chi\{t \cdot \chi^{-1}(y)\}. \tag{27}$$

Hence

$$\chi\{t \cdot \chi^{-1}(x)\} = ax, \tag{28}$$

where a is an arbitrary constant. From (28) we have

$$\chi^{-1}(ax) \cdot \chi^{-1}(1) = \chi^{-1}(a)\chi^{-1}(x). \tag{29}$$

Therefore

$$\chi^{-1}(x) = \chi^{-1}(1) \cdot x^c,$$

where c is an arbitrary constant. Similarly, if instead of (26) we assume

$$f(tx, ty) = f(x, y)$$

or

$$f(t + x, t + y) = f(x, y),$$

we find respectively the relations

$$\chi^{-1}\{\chi(tx) - \chi(ty)\} = \chi^{-1}\{\chi(x) - \chi(y)\}$$

or

$$\chi^{-1}\{\chi(t + x) - \chi(t + y)\} = \chi^{-1}\{\chi(x) - \chi(y)\},$$

which yield,‡ essentially, well known functional relations due to Cauchy.

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* Cf. also Dickson, *Transactions*, vol. 4 (1903), pp. 14, 17.
† Cf. Bourlet's "additive transmutation," l. c., pp. 136, 137, 141, etc.
‡ Cf. Bourlet, l. c., p. 143 (8).

SHORTER NOTICES.

Leçons sur le Calcul des Variations. Par J. HADAMARD. Recueillies par M. FRÉCHET. Tome Premier. Paris, Hermann, 1910. 8vo. viii+520 pp. 18 francs.

A NEW aspect is given to the calculus of variations as it is presented by Hadamard in his emphasis upon the functional character of the whole problem. While this aspect of the subject has been mentioned by several writers before Hadamard's treatise appeared, it is an innovation in a formal treatment of the entire subject. The ideas of the functional calculus do not appear on the surface of things in a manner offensive to the traditions, but they are subconsciously present throughout the book, as is explained by Hadamard himself in the preface; and they crop out at the surface occasionally.

Briefly stated, any line integral between fixed end points, for example, is a function of the path of integration, in the sense that when that path is given, the integral is determined. From this standpoint, a problem of the calculus of variations is a problem in finding the maximum or minimum values of a functional expression, the independent variable being a *curve*. Of course, we wish also to determine the value of the independent variable—that is the curve—for which the extreme value occurs.

Since the book is now well known to all who are interested in the calculus of variations, this review must be a retrospect and an appreciation rather than an analysis. Much as the reviewer regrets the obstacles which have prevented an earlier review, this changed viewpoint is not without advantages.

Hadamard at once commanded respectful attention everywhere; an immediate review would not have increased nor diminished the desire for instant examination of the work. The reception the book has received throughout the world fully justifies this sweeping statement. Thus Carathéodory in the *Bulletin des Sciences Mathématiques* (volume 34, pages 124–142) calls it an epoch-making work (un livre qui marquera une date dans l'histoire du sujet). I will content myself with noting briefly the plan of organization and what appear to be the guiding motives.

The book proper is preceded by an introduction on extrema of functions and on the properties of differential equations, and is followed by an appendix on implicit functions. These treatments are auxiliary and consider only questions necessary for the developments of the book itself.

The body of the work is divided into three main divisions, called books. In the first book, the task attempted is only the setting of the problem. This is done, to be sure, in a fashion not at all naïve. In fact, the difficulties peculiar to the subject, such as the familiar distinction between strong and weak extrema, the generalized notion of neighborhood, and so on, are fundamentally interesting from the standpoint of the functional calculus; every point of this character is discussed in full, and in a most interesting manner.

In the second book, the conditions of the first order—those associated commonly with the first variation—are discussed, together with generalizations of the problem. Here again, questions of peculiar fundamental difficulty are always discussed carefully as a consideration of the first moment. Thus the disagreement between the problem stated in homogeneous form and that stated in non-homogeneous form, which is not always clearly recognized, is made vividly clear. The preliminary Euler condition, the first variation, the fundamental lemma, conjugate points, transversals, and so on, are discussed, usually without great variation from methods now grown classical.

Especial emphasis is laid on practical applications, too often much neglected. Attention is called particularly to No. 140, page 151, and to No. 314, page 381, in which Hadamard sets forth the claim that the methods of mechanics and those of the treatment of geodetic lines by Darboux not only constitute a special case but practically forecast in a trustworthy manner the entire theory. Not only these broad claims but also profuse individual problems characterize the entire book as one of more than usual direct practical bearing.

In the later chapters of this second book are considered many generalizations of the first problem: the case of several variables, the case on a surface, the case of variable endpoints, the isoperimetric problem, and a detailed account of Mayer's problem.

This last—the problem of Mayer—is a second instance of a treatment of functional expressions, the first instance being

the ordinary problem of the calculus of variations. It is used as a point of departure for a brief but illuminating chapter on the functional calculus, which is the real motive of the entire book, as explained above. The ideas of Volterra are first outlined, including the notion of a derivative of a functional expression. Then the concept of a linear functional expression is introduced, and some of the results of Hadamard, Fréchet and others are expounded. An interesting discussion of this part is given by Carathéodory in his review mentioned above.

The third book consists of a discussion of the final conditions of all types, for the general case of a "free" extremum. This discussion is more closely traditional, perhaps, though it is by no means slavish in following the methods of previous writers in detail. The second variation, the conditions of Jacobi, the fundamental Weierstrass theory, the methods of Hilbert and Kneser, are discussed fully; and such other problems as the case of discontinuous solutions (Carathéodory) and the existence of an absolute extremum (Osgood, Hilbert) are given in satisfactory completeness.

In all, the treatment is certainly well planned and well balanced. Due emphasis is given to generalizations, but the simplest forms of the problem predominate, and the treatment is therefore not inordinately complicated. The work of many authors is presented in a thoroughly digested form, and in a manner which is at once comprehensive and comprehensible; the student is given a well-rounded view of the entire subject. The special interests of the author are limited to their proper proportions as compared with the work of others.

Hadamard's treatise has already affected the development of the calculus of variations and that of the functional calculus; its influence on future developments should be profound.

E. R. HEDRICK.

Les Principes de l'Analyse mathématique. Par PIERRE BOUTROUX. Exposé historique et critique. Tome premier. Paris, Hermann, 1914. 8vo. xi + 547 pp.

THIS work is designed for those who desire a comprehensive view of mathematics, for the purpose of becoming acquainted with its intrinsic significance, and its historical evolution. The object of the book is to exhibit the facts of mathematics,

rather than the methods of demonstration or discovery, from the most elementary to some of the most complex. It is intended to be serviceable to students of mathematics and equally to students of philosophy who desire to be armed with some knowledge of mathematics.

The fundamental basis of an enterprise of this kind is of course the first thing one wishes to find. This we do not discover explicitly stated in the tome under consideration, but in another place where three chapters of this work are printed under the title: "L'objet et la méthode de l'analyse mathématique"* the purpose becomes more evident, particularly as part of the article cited is from the second tome, not yet published. In a different place† the author states his view rather succinctly, in these terms: "The progress of analysis appears to us, not as a continuous evolution, . . . but as an endless duel in which the human mind, which seeks to subdue rebellious matter, gains more and more the advantage, thanks to its suppleness in continually modifying or enlarging its point of view and in fashioning new weapons." The mind meets a problem or class of problems, and whether it is able to solve them completely or not, it at least constructs an algebra of some kind, with which it is able to reach certain definite conclusions. Thus the unsolvability of algebraic equations of order above the fourth led to the theory of groups, which, though a most happy invention, for the discovery of the logical and algebraical nature of the connections between laws or mathematical functions, is, nevertheless, like the algebra of the sixteenth and seventeenth centuries, of limited power, neither the end nor the center of mathematical analysis. In like manner the theory of functions far from becoming elucidated as the years roll by, has become unfathomable, and we shall never succeed in getting it entirely within the limits of our science. We can set up a stable theory of functions only by restricting (and in an arbitrary way for the most part) our conception of functional correspondence.

We find accordingly that the two books of this tome are called respectively, Statement of the facts, and Construction. Under the first heading one finds: numbers, magnitudes, figures, and combinatory calculus. Under the second are grouped algebraic calculus, calculus of functions, and alge-

* *Revue de Métaphysique et de Morale*, May, 1913.
† Same, January, 1913.

braic geometry. We are told that the next tome will contain analytic geometry, imaginary quantities, series, logic of mathematics, and infinitesimal calculus. In the reference* given we find the reason for these divisions, as follows. The scientist was at first content to gather facts. He looked around him and perceived the harmonious properties of the world of numbers and the world of geometric figures, and also of measurable magnitudes, the synthesis of quantity and figure, the union of arithmetic and geometry. But with the diffusion of algebra he ceased to be contemplative, and became constructive. From simple pieces he built more and more elaborate designs until he became so charmed with his creations that he left to the physicist entirely the application and interpretation of his theories. In the end his principal aim ceased to be the acquisition of new facts but became the perfecting of methods and processes. The ideal of this phase of mathematics is the algebraico-logical synthesis, by which one would start from a few premises and deduce with no further reference to the world of facts the whole of mathematics. In this attempt however the mathematician is turned back upon himself, and his work begins to consist in the more minute analysis of his notions and the discernment of subtilities that previously had escaped his attention. Thus we pass from the algebraico-logical synthesis to analysis in the proper sense of the term. For example, we no longer look upon an ellipse as made up of an infinity of infinitesimal particles called points. An ellipse is a law, the law of a locus. This law is pregnant with the properties of the ellipse and a sufficient analysis of it will bring out all the characteristic properties of the ellipse. From this point of view we find the tangent and the curvature, or the area and the length. This is what we find in the infinitesimal calculus when we leave out the infinity. But the object of the analysis is ever new construction.

"Having mined the simple notions like those of algebraic relations or the classical geometric figures, the mathematician attacks the fundamental principles upon which repose the algebraico-logical construction.

"He seeks for example, how one may present and particularize the axioms of geometry in order to simplify so far as possible those he is forced to admit without demonstration, and to reduce their number to a minimum.

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* *Revue de Métaphysique et de Morale*, May, 1913.

"He analyzes the notion of continuity in order to discover what is essential in it for his purposes, and what hypotheses it is necessary but sufficient to make, regarding an aggregate that is discontinuous, whether of points or of numbers, in order to be able to reason about this aggregate as he does about the continuum.

"Then penetrating to the heart of mathematics, the analyst attacks the general notion of function (law of correspondence between magnitudes) such as he conceives it to be a priori. He searches out how he ought to determine this notion, and in what measure it must be limited, in order to subject the function to his methods of calculation and to represent it quantitatively by algebraic expressions. He sees thus the possibility of generalizing and extending considerably the algebraic technique of his predecessors."

The first tome of this interesting work is pleasant reading. It is quite elementary on the whole and the references to parts of the next tome lead one to suppose that some of the more philosophical notions will appear there. There are some inaccuracies noticeable. For example we remark the misuse of the term *associativity* with regard to involution, page 11. Division by zero, page 48, is left a little hazy as to whether the sign ∞ is intended to be included as a number or not. The term *rationnel* in the last note on page 81, does not seem to be what the author intends to say. On page 137 the series for π as well as the decimal, are inaccurate. On page 138 at the bottom, the 2 in front of each radical should be omitted. On page 139 the continued fraction represents $4/\pi$. The figure on page 249 is badly drawn. On page 331 the formula for the second root, x'', needs a 4. However these are slips, and we need not dwell on them. The fundamentals that come out here and there are the vital part. Such as, page 44, "The world of numbers is for us essentially a class of abstract elements, about which we suppose nothing, save that they are subject to certain definite operations." On page 145, "Every relative number is defined by an ordinary number (rational or irrational, called the absolute value of the relative number) and by a sign $+$ or $-$." On page 275, "Algebra is a rule, or more exactly, the totality of rules, according to which one carries out certain transformations or combinations called algebraic; in general these combinations are those defined by the fundamental operations of arithmetic, but

nothing prevents our imagining others." We have however probably said sufficient to show that there is an interest in the book for everyone. JAMES BYRNIE SHAW.

Principes de la Théorie des Fonctions entières d'Ordre infini. By OTTO BLUMENTHAL. Paris, Gauthier-Villars, 1910. vi + 147 pp.

Entire functions of finite order as well as certain classes of entire functions of infinite order have been treated by a number of mathematicians in recent years. In this volume of the Borel series of monographs on the theory of functions, Blumenthal considers the *general* entire function of infinite order, so that the book forms a natural and satisfactory sequel to Borel's own Leçons sur les Fonctions entières of the same series. The interest of the results obtained lies in their generality rather than in their applicability to special entire functions not before treated. These results are in large measure original with Blumenthal although a similar range of ideas had been earlier developed by Kraft (Dissertation, Göttingen, 1903).

It is by the aid of the notion of function-type (fonction-type) that Blumenthal is enabled to overcome the inherent difficulties of the problems which arise. Let $\nu(x)$ be a function which increases to $+\infty$ with x, and let $\mu(x) \geqq \nu(x)$ be a like function whose rate of increase is governed by an inequality

$$\mu(x') \leqq \mu(x)^{1+\epsilon(x)}, \quad x' = x^{1+\frac{1}{\mu(x)}\epsilon(x)}$$

where $\epsilon(x)$ is a decreasing infinitesimal. Then $\mu(x)$ is a *function-type adjoint to* $\nu(x)$ *and* $\epsilon(x)$ if the inequality

$$\mu(x) \leqq \nu(x)^{1+\delta}$$

holds for any $\delta > 0$ and an infinite number of values of x. If $\nu(x)$ is given it is clear that $\mu(x)$ yields a measure of the increase of $\nu(x)$, and at the same time possesses a certain regularity of increase (croissance typique).

The fundamental theorem concerning function-types is that corresponding to any given $\nu(x)$ a function-type $\mu(x)$ adjoint to $\nu(x)$ and some infinitesimal $\epsilon(x)$ may be found. The proof first given by Blumenthal (pages 24–31) contains an error. It is assumed that for any given increasing function $w(x)$ an infinitesimal $\epsilon(x)$ (not the $\epsilon(x)$ of the th orem) can be found such

that both $\epsilon(x) \log x$ and $\epsilon(x) \log w(x)$ are increasing functions. But if $\log \log w(x)$ is made to increase by only a finite amount outside of a set of intervals $(x_i, x_i + \delta_i)$, $i = 1, 2, \cdots$, and if these intervals be so chosen that the series

$$\frac{\delta_1}{x_1} + \frac{\delta_2}{x_2} + \cdots$$

converges, it is easy to see that $\epsilon(x)$ cannot approach zero. The second proof (note I, pages 117–128) is however devoid of any objection.

With the aid of the concept of function-type Blumenthal generalizes almost all the results known for entire functions of finite order but it is perhaps not desirable that the reviewer make an outline of this material.

Here then is a book which the mathematician who is interested in the theory of the entire function will find worthy of his attention. GEORGE D. BIRKHOFF.

Les Systèmes d'Equations linéaires a une Infinité d'Inconnues. Par FRÉDÉRIC RIESZ. Paris, Gauthier-Villars, 1913. vi + 182 pages. 6.50 fr.

THIS little book belongs to the collection of monographs on the theory of functions published under the general direction of M. Emile Borel. It deserves the high praise of being pronounced worthy a place in this excellent series.

The purpose of the volume is to give a rapid exposition of the fundamental ideas, of the methods and of the principal results in the theory of linear equations with an infinite number of variables—a theory which is due almost entirely to contemporary mathematicians.

An introduction to the subject is made through a chapter (Chapter I, pages 1–20) devoted to the beginnings of the theory. The method of undetermined coefficients was the first; but this method is not characteristic of the subject. Next comes the work of Fourier, who introduced an important general principle in connection with a special type of problem: naturally, the work of Fourier does not meet the modern requirements of rigor. Mention is also made of the papers of Fürstenau and Kötteritzsch; these are said to have been without importance in the development of the theory. (In this connection see a paper by the reviewer in the *American Journal of Mathematics* for January, 1914.) Finally, an account of

the work of Appell is given in the latter part of the first chapter.

Chapter II (pages 21–41) begins with a brief account of the ideas introduced by the American astronomer, Dr. G. W. Hill, in his very important memoir on the motion of the lunar perigee. This memoir was the starting point for the modern theory of infinite determinants and of linear equations with an infinite number of variables. Poincaré first supplied the convergence proofs which are needed to justify the processes used formally by Hill. With the ideas of Hill and Poincaré as a basis, a theory of considerable extent and wide range of applicability has been built up, principally by H. von Koch. An account of the general elements of this theory, in their simplest form and without applications, is given in Chapter II.

Up to this point the author has followed the chronological order, except that (in Chapter I) he has given an account of the work of Appell before that of Hill. He now inverts the chronological order, for greater convenience in exposition, giving an account (in Chapter III, pages 42–77) of Schmidt's far-reaching theory before taking up the earlier work of Hilbert. This theory of Schmidt, the most general hitherto developed, is itself extended by the present author so as to include a somewhat larger range of results than those obtained by Schmidt in his original account.

In connection with this chapter one would do well to read the elegant paper by Bôcher and Brand in the *Annals of Mathematics* (2), 13 (1912): 167–186, where an exposition of the Schmidt theory is given which leaves nothing to be desired from the point of view of the reader who is forming a first acquaintance with the subject. In order to have a deep understanding of the theory the reader should also see the geometric interpretation given to the whole matter in the original presentation by Schmidt. This latter paper is fundamentally so simple and elegant that it is sure to become one of the classics of mathematical literature. It is to be regretted that the book under review does not contain an exposition of the geometric ideas lying at the basis of this theory of Schmidt. By means of these geometric considerations the whole matter receives an illumination which probably can be procured for it in no other way.

Chapters IV (pages 78–121) and V (pages 122–155) are devoted to the theory of Hilbert as expounded by him in his

now famous memoirs on the theory of integral equations and by some of his disciples in subsequent contributions. The first of these two chapters is given to the theory of linear substitutions where the number of variables is infinite. The general theory which is developed from this point of view is applied to the derivation of important results concerning a certain type of system of equations with an infinite number of variables. The second of these two chapters is given to the theory of quadratic forms where the number of variables is infinite and to the application of this theory to that of linear equations.

Chapter VI (pages 156–180) is devoted to certain applications of the general theory developed in the preceding part of the book. It falls into three parts dealing with as many distinct topics; namely, linear differential equations in which the coefficients are expansible in Laurent series, integral equations and trigonometric series.

This little book will serve a useful end by affording a ready introduction to one of the most important and most readily accessible phases of the general theory of functions of an infinite number of variables, a field in which at present there lies out before us a vast domain of unexplored territory—a domain in which the present generation will probably make further important explorations.

R. D. CARMICHAEL.

A General Course of Pure Mathematics from Indices to Solid Analytical Geometry. By ARTHUR L. BOWLEY, Sc.D. Oxford, Clarendon Press, 1913. xii + 272 pp.

It would be difficult to give a better brief account of the contents and the purpose of this book than that supplied by the author in the preface. From his remarks, therefore, we shall quote a few sentences, as follows:

"This book is the result of an attempt to bring within two covers a wide region of pure mathematics. Knowledge is assumed of that part of mathematics usually required for matriculation, namely algebra to simultaneous quadratic equations and the substance of the first four books of Euclid, together with a very slight acquaintance with graphic algebra, mensuration, and solid geometry. From this stage the work is carried forward in algebra to the logarithmic series; in

coordinate geometry to the nature of the general conicoid; in trigonometry to the use of Euler's expressions for the sine and cosine, with a careful treatment of imaginary quantities; in calculus to definite integration and the maxima of a function of n independent variables; together with the pure geometry which is necessary for the other subjects. It has been the intention to include the bulk of the results obtained in pure mathematics which admit of rigid proof of a fairly easy character, and are needed by those who use pure mathematics as an instrument in mechanics, engineering, physics, chemistry, and economics. For this purpose a very great deal that is ordinarily contained in text-books has been thrown aside, and only those theorems and formulas which are of direct practical application or which are necessary to lead to others of direct practical application are retained.

"It has also been the intention to give exact definitions and strict proofs, of a more careful nature than those found in many of the more diffuse and elementary books; only two difficulties have been intentionally glozed over, viz., the nature of continuity and the nature of irrationals."

The considerable number of topics, the discussion of which is brought together in this one book, are treated in separate sections and probably in a larger degree of isolation from each other than most readers would expect in a single volume in which all of them find a place. The exposition, on the whole, is fairly satisfactory; some of the sections are excellent. The section on limits and series is the least satisfactory of all; some of the statements in it are properly characterized as awkward. For examples of these awkward statements the reader may see pages 101, 102, 105, 115.

The book as a whole is a contribution of some value to the pedagogy of that part of the mathematical curriculum with which it is concerned. Some of the controlling ideas in its preparation might well be adapted to the needs of American institutions; but the book itself is probably not well suited to such purposes. R. D. CARMICHAEL.

Démonstration du Théorème de Fermat. Par E. FABRY. Paris, Hermann et Fils, 1913. 22 pp. 1.50 fr.

Using the theory of Kummer as a basis the author undertakes to prove that the equation

$$x^\lambda + y^\lambda + z^\lambda = 0,$$

where λ is an odd prime, cannot be true for any three integers x, y, z prime each to each, provided that

(1) no one of these integers is divisible by λ;

(2) one of them is divisible by λ but not by λ^2.

In the discussion of the first of these results there occurs an essential error which has already been pointed out by Mirimanoff (*Comptes rendus*, Paris, 157: 491–492). Fabry's second result was already known to Sophie Germain and Legendre (see Bachmann's Niedere Zahlentheorie, II, page 467).

R. D. CARMICHAEL.

Vectorial Mechanics. By L. SILBERSTEIN. Macmillan and Company, London, 1913. viii + 197 pp.

THERE are not in English so many books on vector analysis and its applications that we may not welcome another. The Gibbs-Wilson is the most extended and detailed as regards vector analysis itself, but contains illustrations from geometry, mechanics, and physics rather than applications to them; it has therefore too much mathematics and too little connected application to be entirely ideal for the young physicist. Coffin's is more evenly balanced, and may serve almost equally well as an introduction to vector analysis and to vector physics. Heaviside's genial treatment is embedded in his Electromagnetic Theory. Now comes Silberstein with a work which passes as lightly as possible over formal vector analysis and concentrates on theoretical mechanics. This is a useful variety to introduce. The notation is that of Heaviside; heavy type for vectors, no sign for the scalar product, and a prefixed V for the vector product.

A number of minor complaints may well be made. The Macmillan zero, more insignificant than an "o," is bad. It is unfortunate to use Clarendons for lettering a figure, especially when italics are used in the text. And what can be the advantage of making the figures (usually) run with their positive direction clockwise as they appear on the page? Why take the velocity potential φ so that $\mathbf{v} = \nabla\varphi$ instead of $\mathbf{v} = -\nabla\varphi$, especially now that Lamb in his classic Hydrodynamics has decided in favor of the latter choice? There are a number of instances which show that the author, despite varied linguistic accomplishments, does not sense the meaning of common English words—otherwise he would not call the component of a vector a scalar, nor would he speak of the algebraic product

of the tensors of two vectors after defining the tensor as a positive number. And when he speaks of "that $\Delta\sigma$ of which $d\sigma$ is the limit" he is probably more unfortunate than wrong. Indeed the meaning of his text is seldom in question.

The one very bad mistake we have discovered is the treatment of the curl and of Stokes's theorem (pages 33–4). Silberstein attempts an intrinsic proof by considering

$$\psi = \lim_{\Delta S \doteq 0} \frac{\int_{\circ} \mathbf{f} \bullet d\mathbf{r}}{\Delta S}.$$

The scalar function ψ depends on position x, y, z and on α, β, γ, the direction cosines of the normal to a planar element at x, y, z. He then says that ψ may be written as the component of a vector $\mathbf{C}$ along the normal $\mathbf{n}, \psi = \mathbf{C} \bullet \mathbf{n}$. That this does not follow is patent. Indeed if we assume that the limit ψ exists, the only hard thing about Stokes's theorem is precisely to show that ψ may be written as $\mathbf{C} \bullet \mathbf{n}$ with $\mathbf{C}$ dependent on x, y, z but not on α, β, γ. I treated the intrinsic method in an article "Divergence and curl," *American Journal of Science*, volume 23 (1907), pages 214–220, and it has probably been treated by others. We may add that the author uses curl, infinitesimal rotation, and other related ideas so often that before his book is finished he has more than enough material to form a proof of Stokes's theorem.

The first quarter of the Vectorial Mechanics develops the elements of vector analysis, including differentiation and integration, but with no reference to the linear vector function. The student would profit by having at hand a more detailed treatment; but it would be difficult for him to find one where the author seemed so thoroughly to think vectors. The fundamental definition of a vector as a free vector and the relation of the definition to directed quantities in physics is explained fairly well, much better than by Coffin. No distinction is made between axial and polar vector, or between scalars and pseudo-scalars—for which we may be thankful.

Mechanics starts off with d'Alembert's principle, Lagrange's equations, and Hamilton's principle. From this it may be inferred that the reader should have considerable familiarity with mechanics before commencing this book. Next we find the principles of work and energy, center of gravity (momentum), and angular momentum (areas). The treatment of the

motion of a rigid body is very pretty. It is, however, necessary to introduce the notion of the symmetric linear function, and unless the reader looks up the reference to Heaviside or similar source, he may find the chapter a trifle hard. The Poinsot motion yields with great ease to vector methods, and the author makes the most of his analysis.

The chapter on the mechanics of deformable bodies begins with a concise summary of properties of the general linear vector function, passes to strains and infinitesimal displacements, discusses surfaces of discontinuity (Hadamard), and terminates with stress. It would be difficult in so short a space (47 pages) to do the work better. There follows thirty pages on hydrodynamics which contain most of the classical theory as far as general properties are concerned, and a few other things. This, too, is thoroughly good.

To this point we have covered only 170 pages of the text. The brevity is partly a matter of conciseness in style, but largely due to systematically thinking and using vectors. Sixty-nine exercises, a table of cartesian-vector equivalents in parallel column, and an index complete the work. We could only wish for fifty pages more in which the classical electromagnetic theory, including a little crystal optics, should be presented as succinctly as the theory of rigid motion, fluid motion, and elastic media. As it is, however, Silberstein has given us an almost ideal introduction to mathematical vector physics.

EDWIN B. WILSON.

NOTES.

THE July number (volume 15, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "A new principle in the geometry of numbers, with some applications," by H. F. BLICHFELDT; "An application of Severi's theory of a basis to the Kummer and Weddle surfaces," by F. R. SHARPE and C. F. CRAIG; "Transformations of surfaces of Voss," by L. P. EISENHART; "Birational transformations of certain quartic surfaces," by F. R. SHARPE and VIRGIL SNYDER; "One-parameter families of curves in the plane," by G. M. GREEN; "The minimum of a definite integral for unilateral variations in space," by G. A. BLISS and A. L. UNDERHILL; "On a method of comparison for triple-systems," by L. D. CUMMINGS; "An existence theorem

for a certain differential equation of the nth order," by W. R. LONGLEY.

THE July number (volume 36, number 3) of the *American Journal of Mathematics* contains the following papers: "On a certain completely integrable system of linear partial differential equations," by E. J. WILCZYNSKI; "On the connection of an abstract set, with applications to the theory of functions of a general variable," by A. D. PITCHER; "On series of iterated linear fractional functions," by R. D. CARMICHAEL; "The derivative of a function of a surface," by C. A. FISCHER; "Some invariants and covariants of ternary collineations," by H. B. PHILLIPS; "A geometrical application of the theory of the binary quintic," by FLORENCE P. LEWIS.

THE Prince Jablonowski society of Leipzig announces the following prize problem for the year 1916: "To extend the theory of linear functional differential equations in any direction. Particularly desirable is a complete consideration of new special cases." The value of the prize is 1,050 marks. Competing memoirs should be in the hands of the secretary before October 31, 1916.

THE academy of sciences of Paris has awarded its Poncelet prize for 1914 to Dr. H. LEBESGUE, of the University of Paris. The grand prize was not awarded. The Lalande prize in theoretical astronomy was awarded to Dr. M. GUILLAUME of the University of Lyon; the Valz prize was divided between Dr. P. CHEVALIER of Shanghai, and Dr. P. SALET, of the observatory of Paris.

THE following doctorates in mathematics and astronomy were conferred by the French universities during the academic year 1912–1913. A. BLONDEL; "On the theory of tides in a canal; application to the Red Sea" (Toulouse); M. LUIZET; "The Cepheides considered as double stars, with a chart of δ" (Lyon); K. POPOFF: "On the movement of 108 Hecube" (Paris); A. VÉRONNET: "Rotation of a heterogeneous ellipsoid and exact figure of the earth" (Paris); P. J. BROWNE: "The problem of inversion proposed by Abel and its generalizations" (Toulouse); G. COTTY: "Abelian functions and the theory of numbers" (Toulouse); G. FASBINDER: "Dynamics of variable

systems and the rotation of the earth" (Paris); M. GEVREY: "Partial differential equations of parabolic type" (Paris); C. PLATRIER: "The minors of Fredholm's determinant and systems of linear integral equations" (Paris); L. ROCHE: "The wave surface in rotative magnetic polarization and more general phenomena" (Paris); L. ROUYER: "The deformation of quadrics and conjugate surfaces with regard to a quadratic complex" (Paris); J. TROUSSET: "Semi-analytic study of the movement of the eighth satellite of Jupiter" (Paris).

IN memory of HENRI POINCARÉ an international committee has been formed for the purpose of raising a subscription, the proceeds of which, after striking a medal of Poincaré, will be administered by the Paris Academy in encouraging young scientists engaged in the fields to which Poincaré contributed especially, viz., analysis, celestial mechanics, mathematical physics, scientific philosophy. Contributors of 25 to 50 francs will receive a bronze medal, of 50 francs or more a silver medal. Subscriptions should be sent to M. E. Lebon, secretary-treasurer, rue des Ecoles, 4 bis, Paris.

THE following advanced courses in mathematics (elementary courses not included) are offered at the Italian universities during the academic year 1914–1915:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Theory of elasticity, especially elastic vibrations, three hours. —By Professor L. DONATI: Electrodynamics of moving bodies; thermodynamics; radiations; hypothesis of quanta with applications, three hours.—By Professor F. ENRIQUES: Theory of algebraic curves and surfaces, three hours.—By Professor S. PINCHERLE: Elliptic functions; integral equations and systems of linear equations with infinite unknowns, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Equilibrium of elastic bodies, four hours.—By Professor M. DE FRANCHIS: Geometry on algebraic surfaces by transcendental aids, four hours.—By Professor G. PENNACCHIETTI: Hydrodynamics, four hours.—By Professor C. SEVERINI: Theory of analytic functions; theory of permutable functions, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Calculus of variations, four hours.—By Professor G. LORIA: Geometric applications of elliptic functions, three hours.—By Professor O. TEDONE: Geometrical and physical optics, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics; the ancient ages, three hours.—By Professor A. DEL RE: n-dimensional analysis of Grassmann with applications to mechanics of the spaces of constant curvature, four and one half hours.—By Professor R. MARCOLONGO: Equations of dynamics; periodic solutions; asymptotic solutions; restricted problem of the three bodies, three hours.—By Professor D. MONTESANO: Theory of algebraic surfaces and of their linear systems; theory of birational transformations of three-dimensional space, three hours.—By Professor E. PASCAL: The functions of lines and the calculus of variations, three hours.—By Professor L. PINTO: Thermodynamics, three hours.—By Professor G. TORELLI: One—and more—dimensional sets; simple and multiple Lebesgue's integrals; functions of a set; derivation of indefinite integrals, four and one half hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable; integral equations, four hours.—By Professor A. COMESSATTI: Algebraic geometry, three hours.—By Professor P. GARRANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Analytic dynamics; problem of the three bodies, four and one half hours.—By Professor G. RICCI: Absolute differential calculus; potential; elasticity, four hours.—By Professor F. SEVERI: Linear systems of plane curves and rational surfaces, four hours.—By Professor A. TONOLO: Fourier's series; partial differential equations, three hours.—By Professor G. VERONESE: Geometrical applications of the theory of sets, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Functions of a complex variable; entire functions of one or of two variables, three hours.—By Professor M. GEBBIA: Electrodynamics (advanced part), four and one half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one half hours.—By Professor A. VENTURI: Foundations of the modern methods in celestial mechanics following Poincaré; method of Hill for the moon, four and one half hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Geometry on an algebraic curve, and applications to linear systems of plane algebraic curves, three hours.—By Professor E. BOMPIANI: Differential geometry, three hours.—By Professor U. CISOTTI: Mechanics of continuous systems; theory of potential; electricity, three hours.—By Professor F. GERBALDI: Functions of a complex variable; elliptic functions, three hours.—By Professor G. VIVANTI: Integral equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Cremona transformations in the plane and in space, three hours.—By Professor G. BIANCHI: Functions of a complex variable; linear differential equations, four and one half hours.—By Professor U. DINI: Integral equations; linear differential equations in the real field, four and one half hours.—By Professor G. A. MAGGI: Principles of analytic mechanics; theory of potential functions; phenomenological theory of electromagnetic field, four and one half hours.—By Professor P. PIZZETTI: Theory of interpolation; general notions of spherical astronomy; general theory of perturbations, four and one half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Calculus of probability, three hours.—By Professor L. SILLA: Kinematics and mechanisms, three hours.—By Professor V. VOLTERRA: Permutable functions; functional derivative equations with applications, three hours.—Elasticity, three hours.—By ————: Theory of functions; elliptic functions, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Potential functions and hydrodynamics, three hours.—By Professor G. FUBINI: Calculus of variations; Fourier's series; the principle of minimum as application of Fourier's series to the calculus of variations, three hours.—By Professor C. SEGRE: Theory of invariants applied to geometry, three hours.—By Professor C. SOMIGLIANA: Magnetism and electromagnetism, three hours.

THE London mathematical society has awarded the de Morgan medal for 1914 to Professor Sir JOSEPH LARMOR, of Cambridge University.

DR. A. BOHR, of the University of Copenhagen, has been appointed reader in mathematical physics at the University of Manchester.

DR. U. CRUDELI has been appointed docent in mathematical physics at the University of Rome.

PROFESSOR L. TONELLI, of the University of Cagliari, will occupy the chair of analysis at the University of Parma during the present year.

DR. E. BOMPIANI has been appointed docent in analytic geometry at the University of Pavia.

DR. A. COMESSATTI has been appointed docent in geometry at the University of Padua.

DR. E. ROSATI has been appointed docent in projective geometry at the University of Pisa.

DR. M. PICONE has been appointed docent in mathematical analysis at the University of Turin.

DR. A. TONOLO has been appointed docent in mathematical analysis at the University of Padua.

PROFESSORS G. SCHEFFERS and C. JOLLES, of the technical school of Berlin have been named Geheime Regierungs-Räte by the German emperor.

PROFESSOR A. GUTZMER has been elected rector of the University of Halle for the coming academic year.

PROFESSOR A. KORN has been appointed honorary professor at the Charlottenburg technical school.

DR. E. HELLINGER, of the University of Marburg, has been appointed associate professor of mathematics at the University of Frankfort.

PROFESSOR P. STÄCKEL, of the University of Heidelberg, has been elected honorary member of the mathematical-physical society of Budapest.

PROFESSOR D. K. PICKEN, of Victoria College, New Zealand, has been appointed master of Ormand College, University of Melbourne.

THE University of Groningen conferred the honorary degree of doctor of mathematics and physics on Professor E. B. VAN VLECK at its recent tercentenary celebration.

PROFESSOR L. E. DICKSON, of the University of Chicago, will lecture at the University of California during the first term of the present academic year.

PROFESSOR ELIJAH SWIFT, of Princeton University, has been appointed Williams professor of mathematics at the University of Vermont.

DR. M. O. TRIPP has been appointed professor of mathematics at Olivet College.

DR. J. E. ROWE, of Dartmouth College, has been appointed assistant professor of mathematics in the Pennsylvania State College.

PROFESSOR W. C. KRATHWOHL, of Ripon College, has been appointed assistant professor of mathematics in the Armour Institute of Technology.

AT the College of the City of New York, Dr. E. E. WHITFORD has been promoted to an assistant professorship of mathematics. Messrs. G. M. HAYES, C. A. TOUSSAINT, J. A. BREWSTER, and S. A. SCHWARZ have been promoted to instructorships in mathematics.

AT Cornell University, Dr. W. A. HURWITZ has been promoted to an assistant professorship of mathematics.

DR. K. P. WILLIAMS, of Indiana University, has been promoted to an assistant professorship of mathematics.

MR. J. D. ESHLEMAN has been appointed instructor in mathematics at Western Reserve University.

DR. L. E. WILLIAMS has been appointed instructor in mathematics at the Georgia School of Technology.

Miss M. E. Wells, of Mount Holyoke College, has been appointed instructor in mathematics at Oberlin College; Professor Mary E. Sinclair, of Oberlin, is absent on leave during the present academic year.

Dr. Susan R. Benedict has been appointed associate professor of mathematics at Smith College.

At the University of Minnesota, Dr. W. H. Bussey has been promoted to an associate professorship of mathematics. Dr. W. F. Holman and Dr. H. L. Slobin have been promoted to assistant professorships of mathematics.

Professor E. P. R. Duval has returned to his former position as associate professor of mathematics in the University of Oklahoma.

At Columbia University, Professor D. E. Smith will be abroad on leave of absence during the coming academic year.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Borgogelli (G.). Di un più semplice modo per disegnare le proiezioni dei cinque poliedri regolari. 2a edizione. Roma, Manuzio, 1914. 16mo. 27 pp.

Budde (E.). Tensoren und Dyaden im dreidimensionalen Raum. Braunschweig, Vieweg, 1914. M. 6.80

Dickson (L. E.). Elementary theory of equations. New York, Wiley, 1914. 8vo. 5+184 pp. Cloth. $1.75

——. Linear Algebras. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 16.) Cambridge, University Press, 1914. 8vo. 8 + 73 pp. 3s.

Ducla (V.). Démonstration d'un théorème de Fermat. 2e édition, revue et corrigée. Pau, Garet & Haristoy, 1914. 8vo. 30 pp.

Dyck (W. v.). Ueber den Verlauf der Integralkurven einer homogenen Differentialgleichung 1ter Ordnung. (*Abhandlungen der k. bayerischen Akademie der Wissenschaften.*) München, Franz, 1914. 8vo. 49 pp. M. 2.00

Elliott (C.). Models to illustrate the foundations of mathematics. Edinburgh, Lindsay, 1914. 8vo. 8+116 pp. 2s. 6d.

Engel (F.). See Liebmann (H.).

GODFREY (C.) and SIDDONS (A. W.). First steps in the calculus. Cambridge, University Press, 1914. 8vo. 102 pp. Cloth. 1s. 6d.

HACK (F.). Wahrscheinlichkeitsrechnung. Neudruck. (Sammlung Göschen. Nr. 508.) Berlin, Göschen, 1914. 12mo. 123 pp. Cloth. M. 0.90

HELM (G.). Die Grundlehren der höheren Mathematik. Neue verbesserte Ausgabe. Leipzig, Akademische Verlagsgesellschaft, 1914. 8vo. 15+419 pp. Cloth. M. 15.00

JUNKER (F.). Höhere Analysis. 2ter Teil: Integralrechnung. 3te, verbesserte Auflage. Neudruck. (Sammlung Göschen. Nr. 88.) Berlin, Göschen, 1914. 12mo. 190 pp. Cloth. M. 0.90

KÖNIG (J.). Neue Grundlagen der Logik, Arithmetik und Mengenlehre, mit dem Bildnis des Verfassers. Leipzig, Veit, 1914. 8vo. 259 pp. M. 9.00

LAMPE (E.). Mathematik. (Aus "Deutschland unter Kaiser Wilhelm II.") Berlin, Hobbing, 1914.

LIEBMANN (H.) und ENGEL (F.). Die Berührungstransformationen. Geschichte und Invariantentheorie. Zwei Referate der Deutschen Mathematiker-Vereinigung. Leipzig, Teubner, 1914. 8vo. M. 3.00

LINDEMANN (F. und L.). See POINCARÉ (H.).

MACALESTER LOUP (W.). En bijdrage tot de theorie der cyclische oppervlakken en congruenties. (Thèse.) Utrecht, J. van Boekhoven, 1914. 4to. 80 pp.

MORITZ (R. E.). Memorabilia mathematica. The philomath's quotation-book. New York, Macmillan, 1914. 8vo. 9+410 pp. Cloth. $3.00

POINCARÉ (H.). Wissenschaft und Methode. Autorisierte deutsche Ausgabe mit erläuternden Anmerkungen von F. und L. Lindemann. (Wissenschaft und Hypothese. XVII.) Leipzig, Teubner, 1914. 8vo. 6+283 pp. M. 5.00

RAECKE (C.). Allgemeiner Beweis des letzten Fermatschen Satzes. Dresden, Becker, 1914.

REY (J.). Discurso leido en la solemne apertura del curso académico de 1913 à 1914. Oviedo, 1913.

SIDDONS (A. W.). See GODFREY (C.).

SISAM (C. H.). See SNYDER (V.).

SNYDER (V.) and SISAM (C. H.). Analytic geometry of space. New York, Holt, 1914. 12mo. 11 + 289 pp. $2.50

STUDY (E.). Die realistische Weltansicht und die Lehre vom Raume. Geometrie, Anschauung und Erfahrung. (*Die Wissenschaft*, Band 54.) Braunschweig, Vieweg, 1914. 8vo. M. 5.20

TIMERDING (H. E.). See WISSENSCHAFTEN.

VOSS (A.). See WISSENSCHAFTEN.

WILDBRETT (A.). Analytische und projektive Geometrie. 1ter Teil: Analytische Geometrie der Geraden und des Kreises. Elemente der projektiven Geometrie. Nürnberg, Korn, 1914. 8vo. 7+144 pp. Cloth. M. 2.90

WISSENSCHAFTEN, Die mathematischen. (Die Kultur der Gegenwart, Teil III, Abteilung 1.) Lieferung II: A. Voss, Die Beziehungen der Mathematik zur Kultur der Gegenwart; H. E. Timerding, Die Verbreitung mathematischen Wissens und mathematischer Auffassung. Leipzig, Teubner, 1914. 8vo. 6+161 pp.

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). A new algebra. Parts 5 and 6. With answers. London, Macmillan, 1914. 8vo. 276 pp. Cloth. 3s.

BORCHARDT (W. G.) and PERROTT (A. D.). A junior trigonometry. With answers. London, Bell, 1914. 8vo. 258 pp. Cloth. 3s. 6d.

BRUNN (J.). Vierstellige Logarithmen. 2te Auflage. Münster, Aschendorff, 1914. 8vo. 11 pp. M. 0.25

CAMMAN (P.) et RÉBOUIS (A. G.). Compléments de géométrie. Paris, Gigord, 1914. 8vo. 8+463 pp.

CAVAZZONI (L.) e CERCIGNANI (E.). Libro di geometria del piano. 2a edizione. Milano, Albrighi & Segati, 1914. 8vo. 300 pp. L. 2.80

CERCIGNANI (E.). See CAVAZZONI (L.).

CHILD (J. M.). See BARNARD (S.).

DALLE (A.) et WAELE (C. DE). Géométrie plane et éléments de topographie. 4e édition. Namur, Wesmael-Charlier, 1914. 2+372 pp. Cloth. Fr. 2.75

DECKERT (A.). Planimetrie, Stereometrie und Trigonometrie. Breslau, Goerlich, 1914. 8vo. 4+214 pp. M. 3.50

DIESENER (H.). Die Buchstabenrechnung und Algebra. 4te umgearbeitete Auflage von E. Novaczkiewicz. Leipzig, Voigt, 1914. 8vo. 6+275 pp. M. 6.00

DOBBS (W. J.). A school course in geometry, including the elements of trigonometry and mensuration, and an introduction to the methods of co-ordinate geometry. With answers. London, Longmans, 1914. 8vo. 22+427 pp. 4s.

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